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UNIVALENT HARMONIC MAPPINGS OF ANNULI AND A CONJECTURE OF J. C. C. NITSCHE

BY

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ABSTRACT

Let $w = f(z)$ be a univalent harmonic mapping of the annulus $\{\rho \le |z| \le 1\}$ onto the annulus $\{\sigma \le |w| \le 1\}$. It is shown that $\sigma \le 1/(1+(\rho^2/2)(\log\rho)^2)$.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. By a univalent harmonic mapping f of D we shall mean that $f(z) = u(z) + iv(z)$ where u and v are real harmonic in D, and f is injective and sense preserving.

We shall consider the case where D is the annulus $\mathcal{A}_\rho = \{z\colon \rho < |z| < 1\}$ and the univalent harmonic mapping $w = f(z)$ maps $\mathcal{A}_\rho$ onto $\mathcal{A}_\sigma = \{w\colon \sigma < |w| < 1\}$.

In [N], Nitsche considered possible values for $\sigma = \sigma(\rho)$ for a fixed ρ. He showed by means of examples that the values $[0, 2\rho/(1+\rho^2)]$ were all attainable for σ. He also showed that there exists $\sigma_0 = \sigma_0(\rho)$ such that for any such univalent f mapping $\mathcal{A}_\rho$ onto $\mathcal{A}_\sigma$, then

$$\sigma \le \sigma_0(\rho), \tag{1.1}$$

and he raised the question as to whether or not $\sigma_0(\rho) = 2\rho/(1+\rho^2)$ was the sharp bound for (1.1).

Though Nitsche's problem has been mentioned in surveys [BH], [D], [S], it is only recently [L] that a quantitative bound has been given.

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In [L] Lyzzaik proved that if $B(s)$ is the Grötzsch domain conformally equivalent to $\mathcal{A}_\rho$, then

$$\sigma \le s. \tag{1.2}$$

This will be discussed further in §3. In [L], it is conjectured that (1.2) is sharp. In this paper we shall prove an estimate which shows that (1.2) is not sharp.

THEOREM 1.1: *Let f be a univalent harmonic mapping of $\mathcal{A}_\rho$ onto $\mathcal{A}_\sigma$. Then*

$$\sigma = \sigma(\rho) \le \frac{1}{1 + (\rho^2/2)(\log \rho)^2}.$$

We may assume throughout that $f \in C^1(\overline{\mathcal{A}_\rho})$. In fact we may take a proper subannulus $\mathcal{A}$ of $\mathcal{A}_\sigma$ close to $\mathcal{A}_\sigma$ itself, and φ a conformal mapping of $f^{-1}(\mathcal{A})$ onto an annulus $\mathcal{A}_{\rho'} = \{z\colon \rho' < |z| < 1\}$ with $\rho' > \rho$ arbitrarily close to ρ. Further, $f(\varphi^{-1}(z)) \in C^1(\overline{\mathcal{A}_{\rho'}})$ since $\partial f^{-1}(\mathcal{A})$ consists of C^∞ curves. Then $cf(\varphi^{-1}(z))$ for a constant c maps onto $\mathcal{A}_{\sigma'} = \{w\colon \sigma' < |w| < 1\}$ with $\sigma' > \sigma$ arbitrarily close to σ.

2. Proof of Theorem 1

Let $w = f(z)$ be a univalent harmonic mapping of the annulus $\mathcal{A}_\rho$ onto $\mathcal{A}_\sigma$. We may assume that $|z| = 1$ and $|w| = 1$ correspond under f.

We shall write $f(z) = R(z)e^{i\psi(z)}$. Then, a straightforward computation shows that

$$\Delta R = R|\nabla\psi|^2. \tag{2.1}$$

Let $\mathcal{G}(z,\zeta)$ be the Green's function for $\mathcal{A}_\rho$ with pole at ζ, and using (2.1) we write the subharmonic function $R(z)$ as

$$R(z) = -\frac{1}{2\pi}\iint_{\mathcal{A}_\rho} \mathcal{G}(z,\zeta)R(\zeta)|\nabla\psi(\zeta)|^2 dA(\zeta) + H(z), \tag{2.2}$$

where H is the harmonic function having boundary values $R(z)$ on each boundary component. Specifically,

$$H(z) = \sigma + \frac{1-\sigma}{\log(1/\rho)}\log\frac{|z|}{\rho}. \tag{2.3}$$

Let $m(r) = \int_0^{2\pi} R(re^{i\theta})d\theta$. Then, with the assumption that the inner boundaries correspond, the univalence of f requires that

$$m'(\rho) \ge 0. \tag{2.4}$$

Computing $m'(\rho)$ by (2.2) and (2.3) we have

$$m'(\rho) = -\frac{d}{dr}\frac{1}{2\pi}\int_0^{2\pi}\iint\limits_{\mathcal{A}_\rho}\mathcal{G}(re^{i\theta},\zeta)R(\zeta)|\nabla\psi(\zeta)|^2 dA(\zeta)d\theta|_{r=\rho} + \frac{2\pi(1-\sigma)}{\rho\log 1/\rho}. \tag{2.5}$$

The term involving $\mathcal{G}(re^{i\theta},\zeta)$ in (2.5) can be evaluated in a standard way. Briefly, if F is continuous on $\overline{\mathcal{A}_\rho}$ and $u(z)$ is defined by

$$u(z) = \frac{1}{2\pi}\iint\limits_{\mathcal{A}_\rho}\mathcal{G}(z,\zeta)F(\zeta)dA(\zeta), \quad z \in \mathcal{A}_\rho,$$

let

$$v(z) = \frac{1}{2\pi}\iint\limits_{\mathcal{A}_\rho}\nabla_z\mathcal{G}(z,\zeta)F(\zeta)dA(\zeta), \quad z \in \mathcal{A}(\rho) \tag{2.6}$$

where ∇_z is the gradient in the z variable. Since

$$\mathcal{G}(z,\zeta) = \log\frac{1}{|z-\zeta|} + h(z,\zeta)$$

with $h(z,\zeta)$ harmonic in each variable separately, we see that the integral in (2.6) exists. Let $\eta(r)$ be a differentiable function of r for $0 \le r < \infty$ such that $\eta(r) = 0$ for $0 \le r \le 1$, $\eta(r) = 1$ for $r \ge 2$, and $\eta'(r) \le 2$. If for small $\varepsilon > 0$ and fixed $z \in \mathcal{A}_\rho$

$$u_\varepsilon(z) = \frac{1}{2\pi}\iint\limits_{\mathcal{A}_\rho}\mathcal{G}(z,\zeta)F(\zeta)\eta\Big(\frac{|z-\zeta|}{\varepsilon}\Big)dA(\zeta),$$

then

$$|v(z) - \nabla u_\varepsilon| \le \frac{1}{\pi}\iint\limits_{|z-\zeta|\le 2\varepsilon}\Big(\frac{1}{|z-\zeta|} + M_1 + \frac{2}{\varepsilon}\Big(\log\frac{1}{|z-\zeta|} + M_2\Big)\Big)M_3 dA(\zeta),$$

where $M_1 = \max_{|z-\zeta|\le 2\varepsilon}|\nabla_z h|$, $M_2 = \max_{|z-\zeta|\le 2\varepsilon}|h(z,\zeta)|$, and $M_3 = \max_{|z-\zeta|\le 2\varepsilon} F(\zeta)$. Letting $\varepsilon \to 0$ we then have

$$\nabla u(z) = v(z).$$

We shall apply this with $F(\zeta) = R(\zeta)|\nabla\psi|^2$, and with our assumption that $f \in \mathcal{C}^1(\overline{\mathcal{A}_\rho})$ we may apply it up to and including the boundary. Thus, the d/dr in (2.5) can be moved under the integral signs.

Now, for each $\zeta \in \mathcal{A}_\rho$, if we set

$$U(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{G}}{\partial r}(\rho e^{i\theta}, \zeta)d\theta, \tag{2.7}$$

then $U(\zeta)$ is the harmonic function in $\mathcal{A}_\rho$ with boundary values $1/\rho$ on $|\zeta| = \rho$ and 0 on $|\zeta| = 1$. This is simply

$$U(\zeta) = \frac{\log|\zeta|}{\rho\log\rho}. \tag{2.8}$$

Using (2.7) and (2.8) in (2.5) we obtain

$$m'(\rho) = \iint\limits_{\mathcal{A}_\rho} R(\zeta)|\nabla\psi(\zeta)|^2 \frac{\log|\zeta|}{\rho\log(1/\rho)}dA(\zeta) + \frac{2\pi(1-\sigma)}{\rho\log 1/\rho},$$

which with (2.4) gives

$$\iint\limits_{\mathcal{A}_\rho} R(\zeta)|\nabla\psi(\zeta)|^2 \log\frac{1}{|\zeta|}dA(\zeta) \le 2\pi(1-\sigma).$$

Integration by parts yields

$$\int_0^{2\pi}\int_\rho^1\int_\rho^r R(te^{i\theta})|\nabla\psi(te^{i\theta})|^2 tdt\frac{dr}{r}d\theta \le 2\pi(1-\sigma). \tag{2.9}$$

Now,

$$\int_0^{2\pi}\sqrt{R(te^{i\theta})}|\nabla\psi(te^{i\theta})|td\theta \ge 2\pi\rho\sqrt{\sigma},$$

so by the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_0^{2\pi}\int_\rho^r R(te^{i\theta})|\nabla\psi(te^{i\theta})|^2 tdtd\theta \ge& \frac{(2\pi\rho\sqrt{\sigma})^2}{2\pi}\log\frac{r}{\rho}\\ =&2\pi\rho^2\sigma\log\frac{r}{\rho}.\end{aligned}$$

Using this in (2.9) we have

$$\rho^2\sigma\int_\rho^1\log(r/\rho)\frac{dr}{r} \le 1-\sigma.$$

Thus,

$$\frac{\rho^2\sigma}{2}(\log\rho)^2 \le 1-\sigma$$

or

$$\sigma \le \frac{1}{(\rho^2/2)(\log\rho)^2+1}.$$

3. Comparisons of estimates

Let $\mu(s)$ denote the module of the Grötzsch domain $B(s)$ which is the unit disk with the segment $0 \le x \le s$ of the real axis removed. If s is chosen so that $B(s)$ is conformally equivalent to $\mathcal{A}_\rho$, then Lyzzaik [L] proved that $\sigma \le s$. Expanding $\mu(s)$ [LV, pp. 60, 61] near 1 we have

$$\mu(s) = \frac{\pi^2}{4\log\left(4/\sqrt{1-s^2} - \delta(s)\right)}$$

where $\delta(s) \sim \sqrt{1-s^2}$ as $s \to 1^-$.

Using the definition of $B(s)$ we then have

$$\log\frac{1}{\rho} \sim \frac{\pi^2}{2\log(16/(1-s^2))}$$

or

$$s \sim \sqrt{1 - 16\exp\Big(\frac{-\pi^2}{2\log 1/\rho}\Big)} \quad \text{as } \rho \to 1^-. \tag{3.1}$$

Thus, for ρ near 1, Lyzzaik's estimate is $\sigma \le s$ where s satisfies (3.1).

The estimate in Theorem 1.1 is of no value when ρ is small. However, for ρ close to 1 it is easy to see that it is substantially smaller than that given by (3.1). On the other hand, if we let $\tau(\rho) = 2\rho/(1+\rho^2)$, which is the inner radius for the examples of Nitsche, and $\sigma(\rho)$ as in Theorem 1, then $(1-\tau(\rho))/(1-\sigma(\rho)) \to 1$ as $\rho \to 1^-$.

References

[BH] D. Bshouty and W. Hengartner, *Univalent harmonic mappings in the plane*, Annales Universitatis Mariae Curie-Skłodowska, Sectio A **XLVIII 3** (1994), 12–42.

[D] P. Duren, *A survey of harmonic mappings in the plane*, Texas Tech University Mathematics Series **18** (1992), 411–415.

[L] A. Lyzzaik, *The module of the image annuli under univalent harmonic mappings and a conjecture of J. C. C. Nitsche*, preprint.

[LV] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd ed., Springer-Verlag, Berlin, 1973.

[N] J.C.C. Nitsche, *On the modulus of doubly-connected regions under harmonic mappings*, The American Mathematical Monthly **69** (1962), 781–782.

[S] G. Schober, *Planar harmonic mappings*, in *Computational Methods and Function Theory Proceedings, Valparaiso 1989*, Lectures Notes in Mathematics **1435**, Springer-Verlag, Berlin, 1990, pp. 171–176.